

A FIBRE BUNDLE DESCRIPTION OF TEICHMÜLLER THEORY

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1. Introduction

(A) In this paper we prove the theorems which we announced in [14] concerning the diffeomorphism groups of a closed surface, and, in addition, the corresponding theorems for the diffeomorphism groups of the closed non-orientable surfaces. Our method is to construct a certain principal fibre bundle, whose total space is the space of smooth conformal structures of a closed surface, whose base is a Teichmüller space, and whose structural group is a subgroup of the diffeomorphism group of the surface. Our bundle has the further property that its tangent bundle sequence embodies the infinitesimal deformation of structure theory (for surfaces) of Kodaira-Spencer [22].

Set theoretically, the construction of our bundle is a modification of the Ahlfors-Bers development of Teichmüller theory. To show that we have produced a topological fibre bundle, we need a new theorem about the continuity of solutions to Beltrami equations with smooth coefficients (see § 3). We have provided a fairly detailed account of our construction, because even where it closely follows the Ahlfors-Bers developments, certain adjustments are needed. Consequently we believe that the reader will find the paper relatively self-contained. For expositions of Teichmüller theory, and for guides to the literature, we refer to Ahlfors [2], Bers [6], Rauch [26], and Teichmüller [30].

(B) We now formulate precisely our main results. Let X be an oriented smooth (= class C^∞) 2-dimensional manifold which is compact and without boundary. We denote by $\mathbf{D}(X)$ the topological group of all orientation preserving diffeomorphisms of X , endowed with the C^∞ -topology of uniform convergence of differentials of all orders; $\mathbf{D}_0(X)$ is the subgroup consisting of the diffeomorphisms which are homotopic to the identity. (We shall find later that $\mathbf{D}_0(X)$ is the arc component in $\mathbf{D}(X)$ of the neutral element.)

We denote by $\mathbf{M}(X)$ the space of smooth complex structures on X compatible with its given orientation, and give $\mathbf{M}(X)$ the C^∞ -topology. Then (viewing the elements of $\mathbf{M}(X)$ as smooth tensor fields on X) we have a natural action

$$\mathbf{M}(X) \times \mathbf{D}(X) \rightarrow \mathbf{M}(X) .$$

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The following results are established in §§ 5, 6, 8.

Theorem. *Assume that X has genus $g > 1$.*

1. $\mathbf{M}(X)$ is a contractible complex analytic manifold modeled on a Fréchet space.

2. $\mathbf{D}(X)$ acts continuously, effectively, and properly on $\mathbf{M}(X)$.

3. If

$$(1.1) \quad \bar{\Phi}: \mathbf{M}(X) \rightarrow \mathbf{T}(X) = \mathbf{M}(X)/\mathbf{D}_0(X)$$

denotes the indicated quotient map (where $\mathbf{T}(X)$ is given the quotient topology), then (1.1) is a universal principal $\mathbf{D}_0(X)$ -fibre bundle.

4. Let G be the Lie group of automorphisms of the upper half plane. Then $\mathbf{T}(X)$ can be embedded as a real analytic submanifold of G^{2g} . The complex structure of $\mathbf{M}(X)$ induces a complex structure on $\mathbf{T}(X)$, with $\bar{\Phi}$ holomorphic.

$\mathbf{T}(X)$ is the Teichmüller space of the oriented surface X ; its complex structure is the standard one. The quotient group $\mathbf{D}(X)/\mathbf{D}_0(X)$ acts properly discontinuously on $\mathbf{T}(X)$, and its quotient space $\mathbf{R}(X)$ is the Riemann space of moduli of X .

Part 4 of our theorem is known [1], [7], since $\mathbf{T}(X)$ can be identified with the classical Teichmüller space of closed surfaces of genus g .

There are an analogous result for the case $g = 1$ (Theorem 10F) and a suitable statement for the case $g = 0$ (Theorem 9B). We also have a formulation, in the context of conformal structures, for non-orientable surfaces (§ 11).

In broad terms, our proof proceeds by transferring our activities from X to its universal cover, and studying Beltrami's equation there. A technical fact (Theorem 3B) of importance throughout is the continuous dependence of the solution of Beltrami's equation on its coefficients.

(C) Teichmüller's theorem [6] asserts that $\mathbf{T}(X)$ is a cell. Together with the covering homotopy theorem this implies that the fibration (1.1) is topologically trivial. We outline in § 8E an alternative proof of that triviality by constructing a continuous section of $\bar{\Phi}$, based on the existence theorem for harmonic maps [16]. There is a holomorphic section if $g = 1$; but none for $g > 1$ [12].

(D) The next results are interpretations of the development in § 7, in the spirit of Kodaira-Spencer [22] and Weil [32], [33]. We appeal to § 7 for an explanation of the terminology.

Theorem. *Assume that X has genus $g > 1$. Fix any complex structure $J \in \mathbf{M}(X)$.*

1. *The tangent space of $\mathbf{M}(X)$ at J consists of the space of $\bar{\partial}$ -closed 1-forms on X with values in the vector bundle $T^{1,0}(X)$. The kernel of the differential $d\bar{\Phi}(J)$ is identified with the space of such $\bar{\partial}$ -derived 1-forms.*

2. *The tangent space of $\mathbf{T}(X)$ at $\bar{\Phi}(J)$ is given by the cohomology space*

$H^1(X, \Theta)$, where Θ is the sheaf of germs of smooth sections of $T^{1,0}(X) \otimes T^{*0,1}(X)$. $H^1(X, \Theta)$ is conjugate to the space of J -holomorphic quadratic differentials on X .

3. Suppose we represent X (using J) as the quotient of the upper half plane U by a Fuchsian group Γ , acting freely on U . Then the differential of Φ induces an isomorphism of $H^1(X, \Theta)$ onto $H^1(\Gamma, \mathfrak{g})$.

Here $H^1(\Gamma, \mathfrak{g})$ denotes the cohomology space of the discrete group Γ relative to its adjoint representation on the Lie algebra of G . It measures the infinitesimal deformations of Γ in G .

(E) The following is a purely topological conclusion; it assembles results from §§ 8–11.

Let X be a closed surface. We extend the notation $\mathbf{D}(X)$ to non-orientable X , defining it for that case as the topological group of all diffeomorphisms.

Corollary.

1. If X is the sphere or projective plane, then $\mathbf{D}(X) = \mathbf{D}_0(X)$ has $SO(3)$ as strong deformation retract.

2. If X is the torus, then $\mathbf{D}_0(X)$ has X as strong deformation retract.

3. If X is the Klein bottle, then $\mathbf{D}_0(X)$ has $SO(2)$ as strong deformation retract.

4. In all other cases $\mathbf{D}_0(X)$ is contractible.

The case of the sphere was first established by Smale [29], using different methods.

Remark. In case 4, it follows that all fibre bundles with structural group $\mathbf{D}_0(X)$ are topologically trivial. In particular, that is true of the bundle over $\mathbf{T}(X)$ with fibre model X , associated with the principal bundle $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X)$ using the natural action of $\mathbf{D}_0(X)$ on X . The total space of that bundle has a natural complex structure, making it a holomorphic family of compact Riemann surfaces [3], [22].

Remark. The spaces $\mathbf{D}(X)$, $\mathbf{D}_0(X)$, $\mathbf{M}(X)$, and $\mathbf{T}(X)$ are absolute neighborhood retracts, being metrizable manifolds modeled on Fréchet spaces. In particular, they are absolute retracts if they are contractible.

(F) **Remark.** Theorems 1C and 1D suggest the form of a global deformation theory for structures on closed manifolds X : Start with a smooth bundle $\gamma: V \rightarrow X$ associated with the principal bundle of X . Then the space $\mathcal{C}(\gamma)$ of C^r -sections ($0 \leq r \leq \infty$) of γ is an infinite dimensional manifold. Specify a subgroup \mathcal{S} of $\mathbf{D}(X)$; then \mathcal{S} acts continuously on $\mathcal{C}(\gamma)$, and we can form the quotient space $\mathbf{T}(\gamma; \mathcal{S})$. In a large variety of cases the differential of the quotient map $\Phi: \mathcal{C}(\gamma) \rightarrow \mathbf{T}(\gamma; \mathcal{S})$ determines the infinitesimal deformation theory of Kodaira-Spencer.

2. Complex structures

(A) A complex structure on the oriented vector space \mathbf{R}^2 is an endomorphism J of square $-I$ such that $\det(v, Jv) > 0$ for $v \in \mathbf{R}^2$. The space \mathbf{M}

of all such structures is the homogeneous space $GL^+(\mathbf{R}^2)/GL(\mathbf{C}^1)$. Here $GL^+(\mathbf{R}^2)$ is the group of real 2×2 matrices with positive determinant, and $GL(\mathbf{C}^1)$ is the multiplicative group of non-zero complex numbers, embedded in $GL^+(\mathbf{R}^2)$ by

$$a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

On the other hand, if we write $a + ib$ in the form $r \exp i\theta$, $r > 0$, we can identify $GL(\mathbf{C}^1)$ with $GL^+(\mathbf{R}^1) \times SO(\mathbf{R}^2)$, where $SO(\mathbf{R}^2)$ is the rotation subgroup of $GL^+(\mathbf{R}^2)$. The corresponding homogeneous space is the space of conformal structures on \mathbf{R}^2 , and we have the canonical identification

$$(2.1) \quad GL^+(\mathbf{R}^2)/GL(\mathbf{C}^1) = M = GL^+(\mathbf{R}^2)/GL(\mathbf{R}^1) \times SO(\mathbf{R}^2)$$

of the complex and conformal structures on \mathbf{R}^2 . (We recall that a conformal structure on \mathbf{R}^2 is an equivalence class of positive definite quadratic forms on \mathbf{R}^2 , where two such forms are equivalent if they are proportional.)

As is well known, the homogeneous space M can be represented as the open unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ in \mathbf{R}^2 . We do so by associating with each $\mu \in \Delta$ the equivalence class of the quadratic form

$$(2.2) \quad Q(x, y) = |z + \mu\bar{z}|^2, \quad z = x + iy.$$

(B) Let X be an oriented connected smooth ($=C^\infty$) 2-manifold. From its principal $GL^+(\mathbf{R}^2)$ -bundle we construct the associated homogeneous bundle with fibre M . We denote by $\mathbf{M}(X)$ the space of smooth sections of this bundle, endowed with the C^∞ -topology, i.e., the topology of uniform convergence of all differentials on compact subsets of X . The elements of $\mathbf{M}(X)$ are well known to be the almost complex structures on X which are compatible with its orientation. Since X is 2-dimensional, every almost complex structure is integrable, and so $\mathbf{M}(X)$ is the space of complex structures on X [31, Ch. II N°3]. Of course the identification (2.1) means that $\mathbf{M}(X)$ can equally well be considered as the space of conformal structures on X .

3. Beltrami's equation

(A) Let D be a subregion of \mathbf{R}^2 . The Fréchet space $C^\infty(D, \mathbf{C})$ is the vector space of smooth complex-valued functions on D with the C^∞ -topology. The space $\mathbf{M}(D)$ of complex structures on D may be identified with the subset $C^\infty(D, \Delta)$ of $C^\infty(D, \mathbf{C})$ through our identification (2.2) of Δ with M . Explicitly, each $\mu: D \rightarrow \Delta$ induces the conformal ($=$ complex) structure on D represented by

$$(3.1) \quad ds = |dz + \mu(z)d\bar{z}|.$$

We note that the zero function induces the usual complex structure on D .

Suppose that D has the structure (3.1) and C its usual complex structure. Then the map $w: D \rightarrow C$ is holomorphic if and only if it satisfies Beltrami's equation

$$(3.2) \quad w_{\bar{z}} = \mu w_z,$$

where

$$w_z = \frac{1}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right), \quad w_{\bar{z}} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right).$$

(B) Since $|\mu(z)| < 1$ for all $z \in D$, the Beltrami equation (3.2) is elliptic. (3.2) is uniformly elliptic in D if and only if there is a number k such that

$$|\mu(z)| \leq k < 1, \quad z \in D.$$

The theory of uniformly elliptic Beltrami equations is thoroughly developed [3], [4], [10], [24].

Every such equation has a solution which is a diffeomorphism of D onto a region in the plane. If D is the plane C , there is a unique solution of (3.2), denoted by w_μ , which is a diffeomorphism of C onto itself and leaves the points $0, 1, \infty$ fixed. If D is the upper half plane $U = \{z \in C: \text{Im } z > 0\}$, there is a unique solution of (3.2), again denoted by w_μ , which is a homeomorphism of the closure of U onto itself and leaves $0, 1, \infty$ fixed. In both cases w_μ will be called the normalized solution of (3.2).

We shall need the following theorem about the dependence of w_μ on μ . For its proof we refer to the companion paper [15]. (The theorems of our announcement [14] were based on a more primitive version, proved by us somewhat differently, following [10]). In the statement of the theorem, D is either U or C .

Theorem. *For each positive number $k < 1$, the map $\mu \mapsto w_\mu$ is a homeomorphism of the set of $\mu \in \mathbf{M}(D)$ with $\sup \{|\mu(z)| : z \in D\} \leq k$ onto its image in $C^\infty(D, C)$.*

Remark. The construction of homeomorphisms and diffeomorphisms as global solutions of elliptic systems provides a promising tool in topology. For instance,

1) the above theorem implies almost immediately Smale's theorem that the identity component of the diffeomorphism group of the 2-sphere has the rotation group as strong deformation retract—as we shall find in § 9;

2) the homotopy types of the groups of diffeomorphisms of closed surfaces of higher genera can be determined by constructing harmonic maps [16] (diffeomorphic solutions of a second order elliptic system, namely the Euler-Lagrange equation of the energy integral of § 8E below), utilizing the results of [20] and [28]. Further discussion will be given in § 8E.

4. Fuchsian groups

(A) The uniformization theorem says that every simply connected Riemann surface (= surface with complex structure) is conformally equivalent to the Riemann sphere, to \mathbf{C} , or to the upper half plane U (each with its usual complex structure). A complex structure on the surface X induces a complex structure on its universal covering surface \tilde{X} , which is therefore (equivalent to) one of the above.

With four exceptions (X the plane, punctured plane, torus, or sphere), $\tilde{X} = U$, and the cover group Γ is a properly discontinuous group of holomorphic automorphisms of U , acting freely on U . Such a group is called a *Fuchsian group*. (By requiring a Fuchsian group to act freely we are violating standard usage; for our purposes it is convenient to do so).

(B) The group G of all holomorphic automorphisms of U consists of the Möbius transformations

$$Az = (az + b)(cz + d)^{-1}; \quad a, b, c, d \in \mathbf{R}; \quad ad - bc = 1.$$

G is therefore a 3-dimensional Lie group, isomorphic to $SL(\mathbf{R}^2)$ modulo its center. Its Lie algebra \mathfrak{g} is $\mathfrak{sl}(\mathbf{R}^2)$, the algebra of 2×2 real matrices of trace zero. The *adjoint representation* $u \mapsto u^A$ of G on \mathfrak{g} is defined by $u^A = (\text{Ad } A)u$, where $\text{Ad } A: \mathfrak{g} \rightarrow \mathfrak{g}$ is the differential at the identity in G of the map $B \mapsto A^{-1}BA$.

The elements of G are conveniently classified by the positions of their fixed points. An element $A \in G$, not the identity, is called *hyperbolic*, *parabolic*, or *elliptic* according as A has two fixed points in $\mathbf{R} \cup \{\infty\}$, one fixed point in $\mathbf{R} \cup \{\infty\}$ (and no others), or two conjugate non-real fixed points. For us, the hyperbolic and parabolic transformations are of special importance because Γ acts freely and therefore cannot have elliptic elements.

If $A \in G$ is hyperbolic, one of its fixed points is *attractive*, the other *repulsive*. The attractive fixed point z_1 is described by the condition $A^n z \rightarrow z_1$ as $n \rightarrow \infty$ for any $z \in U$. The attractive fixed point of A is the repulsive fixed point of A^{-1} . These assertions are readily verified by noting that every hyperbolic transformation is conjugate in G to a homothetic expansion $z \mapsto kz$ ($k > 1$).

Lemma 1. *If Γ is not cyclic, the centralizer of Γ in G is trivial.*

This classical fact is proved in two steps, both easy. First one proves that two non-trivial elements of G commute if and only if their fixed points coincide. Next one verifies that a discrete subgroup of G whose elements all have the same fixed points is cyclic.

Lemma 2. *If X is compact, Γ consists of hyperbolic transformations. If two elements of Γ have a common fixed point, they commute.*

This lemma is also classical. The first assertion is proved in [6, p. 97]. The second assertion follows from the first, because if two non-commuting

hyperbolic transformations have a (unique) common fixed point, then their commutator is parabolic.

(C) Let X be a compact Riemann surface of genus $g > 1$. As we have seen, there exists a holomorphic covering map $\pi: U \rightarrow X$. π is of course not unique; it may be composed with any element of G . To specify one such π , we mark the surface X by choosing a basepoint $x_0 \in X$ and a canonical system of loops $a_1, \dots, a_g, b_1, \dots, b_g$ generating the fundamental group $\pi_1(X, x_0)$.

Lemma. *For each complex structure $J \in \mathbf{M}(X)$ there is a unique J -holomorphic covering map $\pi: U \rightarrow X$ with Fuchsian cover group Γ such that, for some $z_0 \in \pi^{-1}(x_0)$,*

- 1) the element $A_1 \in \Gamma$ determined by a_1 has its fixed points at 0 and ∞ ,
- 2) the element $B_1 \in \Gamma$ determined by b_1 has its attractive fixed point at 1.

Proof. Given J , choose any holomorphic covering map $\pi_1: U \rightarrow X$ and any $z_1 \in \pi_1^{-1}(x_0)$. Denote the cover group by Γ_1 . Then the elements A_1 and B_1 of Γ_1 determined by a_1 and b_1 do not commute. Thus, by Lemma 2 of §4B, the fixed points of A_1 and the attractive fixed point of B_1 are distinct. Hence there is a unique $A \in G$ which moves the fixed points of A_1 to 0 and ∞ , and the attractive fixed point of B_1 to 1. $\pi = \pi_1 \circ A^{-1}$ is the required covering map.

5. The action of $\mathbf{D}(X)$ on $\mathbf{M}(X)$, $g > 1$

From now until §9, X will be a compact oriented surface of genus $g > 1$, marked as in §4C. In this section we study the action of $\mathbf{D}(X)$ on $\mathbf{M}(X)$. It is convenient to avoid the use of charts on X , employing the uniformization theorem to lift $\mathbf{M}(X)$ and $\mathbf{D}(X)$ to U . We carry out the lifting in §§5A and B.

Many results of this section are true under less stringent assumptions on X . We use the compactness of X only in Propositions 5A and 5D.

(A) Since X is marked, by Lemma 4C each complex structure J in $\mathbf{M}(X)$ determines a smooth covering map $\pi: U \rightarrow X$ whose cover group Γ is Fuchsian. The map π induces a map $\pi^*: \mathbf{M}(X) \rightarrow \mathbf{M}(U)$ whose image we denote by $\mathbf{M}(\Gamma)$; its elements are the Γ -invariant complex structures on U . Recall from §3 that $\mathbf{M}(U)$ is $C^\infty(U, \mathcal{A})$. The uniformization theorem assures that for each $\mu \in \mathbf{M}(U)$ there is a diffeomorphism $w: U \rightarrow w(U) \subset \mathcal{C}$ which satisfies Beltrami's equation (3.2). Moreover, μ is Γ -invariant if and only if $w \circ \gamma$ satisfies (3.2), which happens when and only when

$$(5.1) \quad (\mu \circ \gamma) \overline{\gamma}' / \gamma' = \mu \quad \text{for all } \gamma \in \Gamma .$$

For reasons which will become evident in §7A we denote by $A^1(\Gamma)$ the Fréchet space of all $\mu \in C^\infty(U, \mathcal{C})$ which satisfy (5.1).

Proposition. *$\mathbf{M}(\Gamma)$ is the convex open set in $A^1(\Gamma)$ consisting of those $\mu \in A^1(\Gamma)$ such that $\sup \{|\mu(z)|: z \in U\} < 1$; and $\pi^*: \mathbf{M}(X) \rightarrow \mathbf{M}(\Gamma)$ is a homeomorphism.*

Proof. Since X is compact, Γ has a compact fundamental domain ω . Equation (5.1) shows that $\sup \{|\mu(z)|; z \in U\} = \max \{|\mu(z)|; z \in \omega\}$ for all $\mu \in A^1(\Gamma)$. Thus μ maps U into Δ if and only if that maximum is less than one. The assertion concerning π^* requires no proof.

As an open set in the complex Fréchet space $A^1(\Gamma)$, $\mathbf{M}(\Gamma)$ has a natural complex structure. The map π^* therefore induces a complex structure on $\mathbf{M}(X)$. Any choice of $J \in \mathbf{M}(X)$ leads to the same complex structure on $\mathbf{M}(X)$ because a diffeomorphism $w: U \rightarrow U$ induces a holomorphic automorphism $w^*: \mathbf{M}(\Gamma) \rightarrow \mathbf{M}(w\Gamma w^{-1})$. Thus we obtain the

Corollary. $\mathbf{M}(X)$ is a contractible complex analytic manifold modeled on a Fréchet space.

(B) Let $\mathbf{D}(U)$ be the group of orientation preserving diffeomorphisms of U . As a subset of $C^\infty(U, \mathbb{C})$, $\mathbf{D}(U)$ is metrizable. Furthermore, it is a topological group, by an easy application of Arens' theorem [5]. Let $\mathbf{D}(\Gamma)$ be the normalizer of Γ in $\mathbf{D}(U)$. Then the covering map π induces a continuous epimorphism $\pi_*: \mathbf{D}(\Gamma) \rightarrow \mathbf{D}(X)$ with kernel Γ , given by $\pi_*(f) \circ \pi = \pi \circ f$.

Lemma. π_* is an open map.

Proof. The hyperbolic metric $ds = |z - \bar{z}|^{-1} |dz|$ defines on U a complete Γ -invariant Riemannian structure of constant curvature -4 . Any two points z_1, z_2 in U can be joined by a unique geodesic segment whose length is the hyperbolic distance $\rho(z_1, z_2)$.

Let (g_n) be a sequence in $\mathbf{D}(X)$ converging to the identity 1. Choose z_0 in U and a sequence (f_n) in $\mathbf{D}(\Gamma)$ so that $\pi_*(f_n) = g_n$ and $f_n(z_0) \rightarrow z_0$. The hypothesis on (g_n) means that for each small open set 0 in U there is a sequence (γ_n) in Γ such that $\gamma_n \circ f_n \rightarrow 1$ in $C^\infty(0, \mathbb{C})$. Hence on each compact subset of 0

$$\rho(f_n(z_1), f_n(z_2)) = \rho(\gamma_n(f_n(z_1)), \gamma_n(f_n(z_2))) \leq K\rho(z_1, z_2)$$

for some number K . It follows that the same inequality (with different K) holds on compact subsets of U . Because $f_n(z_0) \rightarrow z_0$, a subsequence (still called (f_n)) converges, uniformly on compact subsets of U , to a map $f: U \rightarrow U$. But $\pi(f(z)) = \lim g_n(\pi(z)) = \pi(z)$. Thus $f \in \Gamma$; in fact $f = 1$ because $f(z_0) = z_0$ and Γ acts freely. We conclude that $f_n \rightarrow 1$ in $\mathbf{D}(\Gamma)$, for in the above convergence $\gamma_n \circ f_n \rightarrow 1$ in $C^\infty(0, \mathbb{C})$, γ_n must be the identity for large n . The lemma is proved.

Corollary. π_* induces an isomorphism between the topological groups $\mathbf{D}(\Gamma)/\Gamma$ and $\mathbf{D}(X)$.

Let $\mathbf{D}_0(\Gamma) = \{f \in \mathbf{D}(\Gamma): f \circ \gamma = \gamma \circ f \text{ for all } \gamma \in \Gamma\}$, the centralizer of Γ in $\mathbf{D}(\Gamma)$. Recall that $\mathbf{D}_0(X) = \{g \in \mathbf{D}(X): g \text{ is homotopic to the identity}\}$.

Proposition. $\pi_*: \mathbf{D}_0(\Gamma) \rightarrow \mathbf{D}_0(X)$ is an isomorphism of topological groups.

Proof. It is well known that $\pi_*(\mathbf{D}_0(\Gamma)) = \mathbf{D}_0(X)$; see for instance [6, pp. 98–100]. We have already noted that the kernel of $\pi_*: \mathbf{D}(\Gamma) \rightarrow \mathbf{D}(X)$ is Γ .

Since $\mathbf{D}_0(\Gamma) \cap \Gamma$, the center of Γ , is trivial by Lemma 1 of §4B, $\pi_*: \mathbf{D}_0(\Gamma) \rightarrow \mathbf{D}_0(X)$ is bijective.

It remains to show that $\pi_*^{-1}: \mathbf{D}_0(X) \rightarrow \mathbf{D}_0(\Gamma)$ is continuous. Given f in $\mathbf{D}_0(\Gamma)$, let (g_n) be a sequence in $\mathbf{D}_0(X)$ converging to $g = \pi_*(f)$. We must prove that $w_n = \pi_*^{-1}(g_n) \rightarrow f$. By the lemma, there is a sequence (f_n) in $\mathbf{D}(\Gamma)$ such that $f_n \rightarrow f$ and $\pi_*(f_n) = g_n$. Now $h_n = f_n \circ w_n^{-1} \in \text{kernel } \pi_* = \Gamma$, and

$$h_n \circ \gamma \circ h_n^{-1} = f_n \circ \gamma \circ f_n^{-1} \rightarrow f \circ \gamma \circ f^{-1} = \gamma$$

for all $\gamma \in \Gamma$. Choose non-commuting elements γ_1 and γ_2 of Γ . For sufficiently large n , h_n commutes with both γ_1 and γ_2 , whence h_n is the identity. (Otherwise the fixed points of h_n would coincide with those of both γ_1 and γ_2 , which is impossible).

(C) The covering map π transfers the natural action (pulling back the complex structure) of $\mathbf{D}(X)$ on $\mathbf{M}(X)$ to an action of $\mathbf{D}(\Gamma)$ on $\mathbf{M}(\Gamma)$, given by

$$(5.2) \quad (\pi^*J) \cdot g = \pi^*(J \cdot \pi_*g) \quad \text{for } g \in \mathbf{D}(\Gamma), J \in \mathbf{M}(X) .$$

Of course (5.2) is the restriction of the natural action of $\mathbf{D}(U)$ on $\mathbf{M}(U)$. That action has a convenient expression when $\mu \in \mathbf{M}(U)$ is of the form $\mu_f = f_{\bar{z}}/f_z$, $f \in \mathbf{D}(U)$. Indeed, $\mu_f = 0 \cdot f$, the pullback by f of the usual complex structure on U . Thus

$$(5.3) \quad \mu_f \cdot g = (0 \cdot f) \cdot g = 0 \cdot (f \circ g) = \mu_{f \circ g} .$$

Each μ in $\mathbf{M}(\Gamma)$ has the form μ_f ; for we may take $f = w_\mu$, the solution of (3.2) introduced in §3B, since Proposition 5A insures that μ is bounded by some $k < 1$.

Proposition.

1. The action $\mathbf{M}(\Gamma) \times \mathbf{D}(\Gamma) \rightarrow \mathbf{M}(\Gamma)$ defined by (5.2) is continuous.
2. The isotropy group of $0 \in \mathbf{M}(\Gamma)$ is $\mathbf{D}(\Gamma) \cap G = N(\Gamma)$, the normalizer of Γ in G .
3. $\Gamma = \{g \in \mathbf{D}(\Gamma): g \text{ acts trivially on } \mathbf{M}(\Gamma)\}$.
4. $\mathbf{D}_0(\Gamma)$ acts freely on $\mathbf{M}(\Gamma)$.

Proof. 1. The continuity of (5.2) follows from general principles. For an alternative proof using (5.3) and Theorem 3B, we observe that each of the following maps is continuous:

$$(\mu, g) \mapsto (w_\mu, g) \mapsto w_\mu \circ g \mapsto \mu \cdot g .$$

2. The isotropy group of $0 \in \mathbf{M}(\Gamma)$ consists of all $g \in \mathbf{D}(\Gamma)$ which are holomorphic automorphisms of U with its usual complex structure; that group is $\mathbf{D}(\Gamma) \cap G$.

3. Since $\mathbf{M}(\Gamma)$ consists of the Γ -invariant complex structures on U , it is evident that Γ acts trivially on $\mathbf{M}(\Gamma)$. Thus Γ is a subgroup of the group Γ_0

of all g which act trivially; by part 2, Γ_0 in turn is a subgroup of G . If $\Gamma_0 \neq \Gamma$, there would exist a fundamental domain ω for Γ and a pair of Γ_0 -equivalent points $z_1, z_2 \in \omega$ with $z_1 \in \text{Int } \omega$. Let μ be a smooth function on $\text{Int } \omega$ which has compact support containing z_1 but not z_2 . Extending the definition of μ to U by (5.1) we obtain an element of $\mathbf{M}(\Gamma)$ which is not Γ_0 -invariant. We conclude that $\Gamma_0 = \Gamma$.

Part 4 is equivalent to the assertion that $\mathbf{D}_0(X)$ acts freely on $\mathbf{M}(X)$, because π_* is an isomorphism on $\mathbf{D}_0(\Gamma)$. Since the complex structure $J \in \mathbf{M}(X)$ corresponding to $0 \in \mathbf{M}(\Gamma)$ was chosen arbitrarily, we need only consider the isotropy group of $0 \in \mathbf{M}(\Gamma)$ relative to $\mathbf{D}_0(\Gamma)$. That group is $\mathbf{D}_0(\Gamma) \cap N(\Gamma)$, the centralizer of Γ in G , which we know to be trivial.

Corollary. *The natural action of $\mathbf{D}(X)$ on $\mathbf{M}(X)$ is continuous and effective. $\mathbf{D}_0(X)$ acts freely.*

(D) Proposition. *$\mathbf{D}(X)$ acts properly on $\mathbf{M}(X)$.*

Proof. The condition of proper action means that the map $\theta: \mathbf{M}(X) \times \mathbf{D}(X) \rightarrow \mathbf{M}(X) \times \mathbf{M}(X)$ defined by $\theta(J, f) = (J, J \cdot f)$ is proper. We shall prove the corresponding assertion in U .

First, let $K \subset \mathbf{M}(\Gamma) \times \mathbf{D}(\Gamma)/\Gamma$ be a closed set, and $((\mu_n, \nu_n))$ a sequence in $\theta(K)$, converging to (μ, ν) . Fix z_0 in U and a compact fundamental domain ω for Γ , and choose a sequence (f_n) in $\mathbf{D}(\Gamma)$ so that $\nu_n = \mu_n \cdot f_n$, $(\mu_n, f_n \Gamma) \in K$, and $z_n = f_n(z_0) \in \omega$.

Let $w_n = w_{\mu_n}$, $w = w_\mu$, and $h = w_\nu$. By Theorem 3B, $w_n \rightarrow w$. Determine a sequence (g_n) in G so that $g_n \circ w_n \circ f_n$ fixes the points $0, 1, \infty$. Then (5.3) and Theorem 3B imply that $g_n \circ w_n \circ f_n \rightarrow h$; in particular, $g_n(w_n(z_n)) \rightarrow h(z_0) \in U$. Since the points $w_n(z_n)$ lie in a compact subset of U , we can pass to a subsequence so that $g_n \rightarrow g \in G$. Then $f_n \rightarrow w^{-1} \circ g \circ h = f \in \mathbf{D}(\Gamma)$. Obviously $(\mu_n, f_n) \rightarrow (\mu, f)$, and $(\mu, \nu) = (\mu, \mu \cdot f)$ is in the image of K . Thus, θ is a closed map.

It remains to prove that $\theta^{-1}(J_1, J_2)$ is compact for any $(J_1, J_2) \in \mathbf{M}(X) \times \mathbf{M}(X)$. We may use $J = J_1$ to determine $\pi: U \rightarrow X$; then (J_1, J_2) corresponds to $(0, \nu) \in \mathbf{M}(\Gamma) \times \mathbf{M}(\Gamma)$. If $\theta(\mu_1, f_1 \Gamma) = \theta(\mu_2, f_2 \Gamma) = (0, \nu)$, then $0 = \mu_1 = \mu_2 = 0 \cdot f_1 \circ f_2^{-1}$, and $f_1 \circ f_2^{-1} \in N(\Gamma)$ by Proposition 5C. We conclude that $\theta^{-1}(0, \nu)$ either is empty or can be mapped bijectively onto $N(\Gamma)/\Gamma$. But $N(\Gamma)/\Gamma$ is a finite group [34, Ch. II].

Corollary 1. *$\mathbf{D}_0(X)$ acts properly on $\mathbf{M}(X)$.*

In fact, every closed subgroup acts properly.

Corollary 2. *The natural action of $\mathbf{D}(X)/\mathbf{D}_0(X)$ on $\mathbf{M}(X)/\mathbf{D}_0(X)$ is properly discontinuous.*

The proposition implies that the action is proper. But $\mathbf{D}(X)/\mathbf{D}_0(X)$ is discrete because $\mathbf{D}_0(X)$, for compact X , is open in $\mathbf{D}(X)$. Hence the corollary.

The group $\mathbf{D}(X)/\mathbf{D}_0(X)$ is the modular group of genus g . The first proof that its action is properly discontinuous was given by Kravetz [23].

6. The map P

To complete the proof that the action of $\mathbf{D}_0(X)$ on $\mathbf{M}(X)$ defines a principal fibre bundle, we need local cross-sections, which are provided by the Bers coordinates on Teichmüller space [1], [7]. To obtain those coordinates we follow the classical path [1], [6], [7], imbedding Teichmüller space as a smooth manifold of dimension $6g - 6$ in G^{2g} , where again G is the real Möbius group. The imbedding is accomplished by a smooth map $P: \mathbf{M}(X) \rightarrow G^{2g}$ which factors through $\mathbf{M}(X)/\mathbf{D}_0(X)$. In § 7 we shall prove that the differential of P establishes an isomorphism between the theories of infinitesimal deformations of complex structures and of Fuchsian groups.

(A) The assumption introduced in § 5, that X is a marked surface of genus $g > 1$, is still in force. We define $P: \mathbf{M}(X) \rightarrow G^{2g}$ by $P(J) = (A_1, B_1, \dots, A_g, B_g)$. Here A_i and B_i are the elements of Γ determined by the loops a_i, b_i , and Γ is the group determined by J as in Lemma 4C. Of course the set $\{A_1, \dots, B_g\}$ generates Γ . In the spirit of [1], [6], we denote by \mathcal{S} the set of points $(A_1, \dots, B_g) \in G^{2g}$ such that

$$(6.1) \quad \text{the product of commutators } \prod_{1 \leq i \leq g} [A_i, B_i] = 1,$$

$$(6.2) \quad \text{the fixed points of } A_g \text{ and } B_g \text{ are real and distinct,}$$

$$(6.3) \quad A_1(0) = 0, \quad A_1(\infty) = \infty, \quad B_1(1) = 1.$$

It is clear that P maps $\mathbf{M}(X)$ into \mathcal{S} .

Proposition. \mathcal{S} is a real analytic submanifold of G^{2g} of dimension $6g - 6$.

Proof. Let N be the set of $(A_1, \dots, B_g) \in G^{2g}$ which satisfy (6.2) and (6.3). It is clear that N is a real analytic $(6g - 3)$ -dimensional submanifold of G^{2g} . The map $\phi: N \rightarrow G$ given by

$$\phi(A_1, \dots, B_g) = \prod_{1 \leq i \leq g} [A_i, B_i]$$

is real analytic, and $\mathcal{S} = \phi^{-1}(1) \subset N$. The proposition will therefore follow from the implicit function theorem as soon as we prove that the differential of ϕ at every $s \in \mathcal{S}$ is surjective.

Choose $s = (A_1, \dots, B_g) \in \mathcal{S}$ and $u, v \in \mathfrak{g}$. Let

$$C(t) = \phi(A_1, \dots, B_{g-1}, A_g \exp tu, B_g \exp tv), \quad t \in \mathbf{R}.$$

An easy calculation gives

$$C(t) = \exp \{t(u^B - u + v - v^A)^{A^{-1}B^{-1}} + o(t)\},$$

where $A = A_g$ and $B = B_g$. Thus

$$(u^B - u + v - v^A)^{A^{-1}B^{-1}}$$

is in the image of the differential $d\phi(s)$, and all we need to prove is the following lemma, which the reader can easily verify.

Lemma. *If $A, B \in G$ have distinct real fixed points, the map*

$$(6.4) \quad (u, v) \mapsto u^B - u + v - v^A$$

from $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is surjective.

Remark. $u^B - u + v - v^A = w$ is the infinitesimal form of the equation $[A, B] = C$ studied by Ahlfors [1, Lemma 3] in a similar context.

(B) Take any $J_0 \in \mathbf{M}(X)$, and let $\pi: U \rightarrow X$ be the covering map determined by J_0 and the marking of X . Then the cover group Γ is generated by $s = P(J_0) \in G^{2g}$. Composing P with the inverse of the map $\pi^*: \mathbf{M}(X) \rightarrow \mathbf{M}(\Gamma)$ produces a map, still called $P: \mathbf{M}(\Gamma) \rightarrow \mathcal{S}$.

Lemma. $P(\mu) = w_\mu \circ s \circ w_\mu^{-1}$ for all $\mu \in \mathbf{M}(\Gamma)$.

Proof. For any μ in $\mathbf{M}(\Gamma)$, $\pi_\mu = \pi \circ w_\mu^{-1}: U \rightarrow X$ is a covering map, holomorphic from U with its usual complex structure to X with the complex structure $(\pi^*)^{-1}\mu$. The cover group $\Gamma_\mu = w_\mu \circ \Gamma \circ w_\mu^{-1}$ is Fuchsian, and the loops a_1 and b_1 on X determine the transformations $w_\mu A_1 w_\mu^{-1}$ and $w_\mu B_1 w_\mu^{-1}$ in Γ_μ . Because w_μ fixes the points 0, 1, and ∞ , π_μ is the cover map determined by Lemma 4C from the marking of X and the complex structure $(\pi^*)^{-1}\mu$, and hence the lemma is proved.

Proposition. $P: \mathbf{M}(\Gamma) \rightarrow \mathcal{S}$ is continuous. The restriction of P to any finite dimensional affine subspace is real analytic. Moreover, the kernel $\text{Ker } dP(0)$ of the differential at 0 consists of all $\nu \in A^1(\Gamma)$ such that

$$(6.5) \quad \dot{\gamma}(\nu)(z) = \lim_{t \rightarrow 0} \frac{\gamma_{t\nu}(z) - \gamma(z)}{t}$$

vanishes for all $z \in U$ and $\gamma \in \Gamma$.

Proof. The continuity of P follows at once from the last lemma and Theorem 3B. For any $\gamma \in \Gamma$ consider the map $\mu \mapsto \gamma_\mu = w_\mu \gamma w_\mu^{-1} \in G$, which is real analytic on finite dimensional subspaces by [4], and whose directional derivative at 0 in the direction ν vanishes if and only if $\dot{\gamma}(\nu)(z)$ vanishes for all $z \in U$. The required real analyticity of P is now obvious, for each component map of P has the form $\mu \mapsto \gamma_\mu$. Furthermore, $\nu \in \text{Ker } dP(0)$ if and only if (6.5) vanishes for all γ in a set of generators of Γ , hence for all γ .

(C) **Lemma.** $P(J_0) = P(J_1)$ if and only if J_0 and J_1 are $\mathbf{D}_0(X)$ -equivalent.

Proof. We shall prove that $P(0) = P(\mu)$, $\mu \in \mathbf{M}(\Gamma)$, if and only if 0 and μ are $\mathbf{D}_0(\Gamma)$ -equivalent. By Lemma 6B, $P(0) = P(\mu)$ if and only if $w_\mu \in \mathbf{D}_0(\Gamma)$. But 0 and μ are $\mathbf{D}_0(\Gamma)$ -equivalent if and only if $\mu = \mu_f$ for some $f \in \mathbf{D}_0(\Gamma)$. That f can only be w_μ . In fact $A_1 = f \circ A_1 \circ f^{-1}$ and $w_\mu \circ A_1 \circ w_\mu^{-1}$ both fix 0 and ∞ , while $f \circ B_1 \circ f^{-1} (= B_1)$ and $w_\mu \circ B_1 \circ w_\mu^{-1}$ both have the attractive fixed point 1. Thus, $g = f \circ w_\mu^{-1}$ leaves 1 fixed and maps the set $\{0, \infty\}$ on itself; this implies g is the identity and $f = w_\mu$, because $g \in G$.

7. The infinitesimal theory

Here we investigate the connection between the global space of complex structures on X , described by $\mathbf{M}(X)/\mathbf{D}_0(X)$, and the theory of infinitesimal variations of complex structures, measured by appropriate cohomology spaces. There is also a connection with the theory of infinitesimal deformations of Fuchsian groups. In fact, the cohomology spaces associated with those two theories are isomorphic, the isomorphism being given by the differential of P . In a sense, $P: \mathbf{M}(X) \rightarrow \mathcal{S}$ is the envelope of the cohomology isomorphisms. Our point of view in this section has been influenced by Weil's paper [32].

(A) A complex structure J_0 on X defines on each tangent vector space $T_x(X)$ an endomorphism $J_0(x)$ of square $= -I$. This extends to a complex endomorphism $J_0(x)$ of $CT_x(X) = C \otimes_R T_x(X)$; that space has the direct sum decomposition $T_x^{1,0} \oplus T_x^{0,1}$, where $T_x^{1,0}$ (resp. $T_x^{0,1}$) is the image of the projection operator $\frac{1}{2}(I - iJ_0(x))$ (resp. $\frac{1}{2}(I + iJ_0(x))$). This induces a similar decomposition on all tensor products of $CT_x(X)$ and its dual space $CT_x(X)^*$.

Let A^p be the vector space of smooth differential forms on X of type (o, p) with values in the vector bundle $T^{1,0}(X)$. The $(0, 1)$ -component $\bar{\partial}$ of the exterior differential maps A^p into A^{p+1} . Following Kodaira-Spencer [22], let Θ denote the sheaf of germs of smooth sections of $T^{1,0} \otimes T^{*0,1}$. The $\bar{\partial}$ -cohomology group $H^1(X, \Theta)$ measures the infinitesimal variations of J_0 ; because $\bar{\partial}$ is zero on A^1 , $H^1(X, \Theta) = A^1/\bar{\partial}A^0$.

Remark. The complex structure J_0 identifies the vector space of smooth real vector fields on X with A^0 . Indeed, suppose $v \in C^\infty(CT(X))$ is expressed as $v = v^{1,0} + v^{0,1}$; then v is real if and only if $(v^{1,0})^- = v^{0,1}$.

(B) Once more we pass to the universal covering space U by the holomorphic covering map π . In the notation of § 5A, the space A^0 lifts to

$$A^0(\Gamma) = \{f \in C^\infty(U, C) : (f \circ \gamma)' / \gamma' = f \text{ for all } \gamma \in \Gamma\} ;$$

the space A^1 lifts to $A^1(\Gamma)$. Of course, with this interpretation $\bar{\partial}f = f_{\bar{z}}$.

Let $Q(\Gamma)$ be the lift of the vector space $H^0(X, T^{*1,0} \odot T^{*1,0})$ of holomorphic quadratic differentials; then $Q(\Gamma)$ consists of the holomorphic functions φ on U which satisfy

$$(\varphi \circ \gamma)(\gamma')^2 = \varphi \quad \text{for all } \gamma \in \Gamma .$$

The vector spaces $H^0(X, T^{*1,0} \odot T^{*1,0})$ and $H^1(X, \Theta)$ are conjugate. This special case of Serre's duality theorem [27] - also known as Teichmüller's Lemma - is a consequence of the next

Proposition. $\text{Ker } dP(0) = \bar{\partial}A^0(\Gamma) = Q(\Gamma)^\perp$, where

$$Q(\Gamma)^\perp = \left\{ \nu \in A^1(\Gamma) : \int_X \nu \varphi d\bar{z} \wedge dz = 0 \quad \text{for all } \varphi \in Q(\Gamma) \right\} .$$

Proof. Let $\nu \in \text{Ker } dP(0)$. By [3, p. 138], [1],

$$\dot{\gamma}(\nu) = f \circ \gamma - \gamma' f, \quad \text{where } f_{\bar{z}} = \nu.$$

But $\dot{\gamma}(\nu)$ vanishes for all $\gamma \in \Gamma$ by Proposition 6B. Therefore $f \in A^0(\Gamma)$, and we have proved

$$(7.1) \quad \text{Ker } dP(0) \subset \bar{\delta}A^0(\Gamma).$$

Next, take any $f \in A^0(\Gamma)$ and set $\nu = f_{\bar{z}}$. Then for each $\varphi \in Q(\Gamma)$, $\omega = f\varphi dz$ is a 1-form on X . By Stokes' theorem

$$\int_{\bar{X}} \nu \varphi d\bar{z} \wedge dz = \int_{\bar{X}} d\omega = 0.$$

Thus

$$(7.2) \quad \bar{\delta}A^0(\Gamma) \subset Q(\Gamma)^\perp.$$

From (7.1) and (7.2), $\text{codim Ker } dP(0) \geq \dim Q(\Gamma)$, which is $6g - 6$ by the Riemann-Roch theorem. Since the kernel of $dP(0)$ has codimension no greater than $6g - 6$, the dimension of \mathcal{S} , we conclude that $Q(\Gamma)^\perp = \text{Ker } dP(0)$.

(C) We now define $H^1(\Gamma, \mathfrak{g})$, the 1-dimensional cohomology space of Γ relative to the adjoint representation, as follows. A 1-cocycle is a map $f: \Gamma \rightarrow \mathfrak{g}$ satisfying

$$(7.3) \quad f(\gamma_1 \circ \gamma_2) = f(\gamma_1)^{\gamma_2} + f(\gamma_2),$$

and the coboundary δu of $u \in \mathfrak{g}$ is the 1-cocycle

$$\delta u(\gamma) = w - u.$$

Thus $H^1(\Gamma, \mathfrak{g})$ is the quotient vector space of cocycles modulo coboundaries, which measures the infinitesimal deformations of Γ in G [32], [33].

Proposition. *The tangent space $T_s(\mathcal{S}) = H^1(\Gamma, \mathfrak{g})$, where $s = P(0) \in \mathcal{S}$.*

Proof. We construct a linear map $L: T_s(\mathcal{S}) \rightarrow H^1(\Gamma, \mathfrak{g})$ as follows: From each smooth curve $c: (-1, 1) \rightarrow \mathcal{S}$ with $c(0) = s$, construct a curve of homomorphisms $\varphi_t: \Gamma \rightarrow G$ by setting $c(t) = (\varphi_t(A_1), \dots, \varphi_t(B_g))$. The curve φ_t gives rise to a 1-cocycle $f: \Gamma \rightarrow \mathfrak{g}$ in the usual way:

$$\gamma^{-1}\varphi_t(\gamma) = \exp(tf(\gamma) + \circ(t)) \quad \text{for all } \gamma \in \Gamma.$$

Clearly f depends only on the tangent vector $c'(0) \in T_s(\mathcal{S})$. We define $L(c'(0))$ to be the image of f in $H^1(\Gamma, \mathfrak{g})$.

We show next that L is injective. Suppose that the cocycle f determined by φ_t is a coboundary: $f(\gamma) = w - u$. Then the curves $\varphi_t(\gamma)$ and $\exp(tu)_\gamma \exp(-tu)$

are tangent at $t = 0$ for all $\gamma \in \Gamma$. Because A_1 and $\varphi_t(A_1)$ fix 0 and ∞ , u is a diagonal matrix. Because B_1 and $\varphi_t(B_1)$ leave 1 fixed and B_1 has distinct fixed points, u is the zero matrix.

It remains to show that L is surjective, a consequence of the

Lemma. $\dim H^1(\Gamma, \mathfrak{g}) = 6g - 6 = \dim \mathcal{S}$.

Proof. We already know that $\dim H^1(\Gamma, \mathfrak{g}) \geq \dim \mathcal{S} = 6g - 6$ because $L: T_s(\mathcal{S}) \rightarrow H^1(\Gamma, \mathfrak{g})$ is injective. On the other hand, it is easy to verify, using (7.3), (6.1) and Lemma 6A, that the space of 1-cocycles has dimension not exceeding $6g - 3$. Finally, $u \mapsto \delta u$ maps \mathfrak{g} injectively to the space of 1-coboundaries: if $u^{A_1} = u$, then u is diagonal; if also $u^{B_1} = u$, then u is the zero matrix.

(D) **Theorem.** *The differential*

$$dP(0): A^1(\Gamma) \rightarrow H^1(\Gamma, \mathfrak{g}) = T_s(\mathcal{S})$$

of $P: \mathbf{M}(\Gamma) \rightarrow \mathcal{S}$ at 0 induces an isomorphism $H^1(X, \Theta) \rightarrow H^1(\Gamma, \mathfrak{g})$.

Proof. Proposition 7B says that $\text{Ker } dP(0) = \bar{\partial}A^0(\Gamma)$, and that the real dimension of $H^1(X, \Theta) = \dim Q(\Gamma) = 6g - 6$.

8. The Teichmüller space $T(X)$, $g > 1$

(A) **Proposition.** $P: \mathbf{M}(X) \rightarrow \mathcal{S}$ is an open map with local sections; i.e., for each $s \in P(\mathbf{M}(X))$ there exist a neighborhood N of s in \mathcal{S} and a real analytic map $f: N \rightarrow \mathbf{M}(X)$ with $P \circ f$ as the identity on N .

Proof. Because J_0 was chosen arbitrarily in § 6B, we need only show that the map $P: \mathbf{M}(\Gamma) \rightarrow \mathcal{S}$ is open at the origin and has a local section $f: N \rightarrow \mathbf{M}(\Gamma)$ defined in a neighborhood N of $s = P(0)$. This is immediate from Propositions 7B, 7C, and the implicit function theorem.

Recall that *Teichmüller's space* $\mathbf{T}(X)$ is the quotient $\mathbf{M}(X)/\mathbf{D}_0(X)$ with quotient topology. In view of Lemma 6C, Proposition 8A has the immediate

Corollary. $P: \mathbf{M}(X) \rightarrow \mathcal{S}$ has the form $P = h \circ \Phi$, where $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X)$ is the quotient map, and $h: \mathbf{T}(X) \rightarrow P(\mathbf{M}(X))$ is a homeomorphism.

Remark. Map $Q(\Gamma)$ into $A^1(\Gamma)$ by $\varphi \mapsto \bar{\varphi}\lambda^{-2}$, where $\lambda(z)|dz|$ is the hyperbolic metric on U , and denote the image by $\mathcal{H}^1(\Gamma)$. Proposition 7B implies that $A^1(\Gamma)$ is the direct sum of $\text{Ker } dP(0)$ and $\mathcal{H}^1(\Gamma)$; this can be viewed as a case of Hodge's theorem. Hence $\mu \mapsto P_\mu$ is a diffeomorphism from a neighborhood of the origin in $\mathcal{H}^1(\Gamma)$ to an open set in \mathcal{S} , and the set of all such diffeomorphisms provides complex local coordinate charts, the *Bers coordinates*, on $P(\mathbf{M}(X))$. These charts define a complex analytic structure [7] which is the quotient by P of the complex analytic structure of $\mathbf{M}(X)$ defined in § 5A. Each Bers coordinate chart can be extended (uniquely) to a global

coordinate chart f , which is a holomorphic homeomorphism of $P(\mathbf{M}(X))$ onto an open subset of $\mathcal{H}^1(\Gamma)$. The restriction of f to $f^{-1}(\mathbf{M}(\Gamma))$ is a local section of P , the Ahlfors-Weill section [14], and the image of f is a bounded domain of holomorphy in $\mathcal{H}^1(\Gamma)$ [7], [9].

(B) A principal fibre bundle is determined by a continuous action of a topological group on a space, which is free, proper, and locally trivial [21]; the local triviality amounts to the existence of local sections of the quotient map.

Theorem. *The quotient map $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X)$ defines a universal principal fibre bundle with structure group $\mathbf{D}_0(X)$.*

Proof. The theorem consolidates the results of §§ 5C, 5D, and 8A. The bundle is universal because $\mathbf{M}(X)$ is contractible by Corollary 5A.

(C) **Teichmüller's Theorem.** *$\mathbf{T}(X)$ is homeomorphic to \mathbf{R}^{6g-6} .*

We refer to [6] for a particularly simple proof.

Corollary 1. *The bundle $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X)$ is topologically trivial.*

Proof. By Teichmüller's theorem there is a map $g: \mathbf{T}(X) \times [0, 1] \rightarrow \mathbf{T}(X)$ with $g(\tau, 0) = \tau_0$ and $g(\tau, 1) = \tau$. By the covering homotopy theorem there is a map $f: \mathbf{T}(X) \times [0, 1] \rightarrow \mathbf{M}(X)$, which covers g . $\sigma(\tau) = f(\tau, 1)$ defines a section of the map Φ .

Corollary 2. *$\mathbf{M}(X)$ is homeomorphic to $\mathbf{T}(X) \times \mathbf{D}_0(X)$. In particular, $\mathbf{D}_0(X)$ is contractible.*

Proof. Let $\sigma: \mathbf{T}(X) \rightarrow \mathbf{M}(X)$ be any section of Φ . Then $(\tau, g) \mapsto \sigma(\tau) \cdot g$ is a homeomorphism from $\mathbf{T}(X) \times \mathbf{D}_0(X)$ to $\mathbf{M}(X)$.

(D) **Remarks.**

1. Recently M. E. Hamstrom [19] has computed the homotopy groups of the homeomorphism group $\mathcal{H}(X)$ (a topological group with compact-open topology) of any compact surface X with or without boundary. Comparison of her results with ours shows that in every case $\mathcal{H}_0(X)$ and $\mathbf{D}_0(X)$ have the same homotopy groups. It is reasonable to guess that the identity map $i: \mathbf{D}_0(X) \rightarrow \mathcal{H}_0(X)$ is a homotopy equivalence, which could be established if it were true that $\mathcal{H}_0(X)$ is an absolute neighborhood retract. It is not known whether $\mathcal{H}_0(X)$ enjoys the last property, although it is a locally contractible metrizable group.

2. Recall that $\mathbf{D}_0(X)$ consists of all $f \in \mathbf{D}(X)$ homotopic to the identity. In the topological category, R. Baer's theorem [17] states that homotopic homeomorphisms of X are isotopic. The fact that $\mathbf{D}_0(X)$ is connected (by Corollary 2 above) gives Baer's theorem in the smooth category.

3. Corollary 1, the contractibility of $\mathbf{T}(X)$, and the contractibility of $\mathbf{D}_0(X)$ are equivalent properties. A. Grothendieck conjectured such a relationship [18], emphasizing the importance of a topological proof that $\mathbf{D}_0(X)$ is contractible. We sketch an analytical proof (therefore violating the spirit of Grothendieck's conjecture) in § 8E, and construct an explicit section of Φ .

4. By Remark 8A, $\mathbf{T}(X)$ has a complex structure such that $\Phi: \mathbf{M}(X)$

$\rightarrow \mathbf{T}(X)$ is holomorphic. Moreover, $\mathbf{T}(X)$ is a Stein manifold [9]. Since the bundle $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X)$ is topologically trivial, one might ask whether it is holomorphically trivial. The answer is no; there are no holomorphic cross-sections of Φ [12]. By contrast, in § 10 we shall define a holomorphic section of Φ when $g = 1$.

5. In the work of Ahlfors and Bers [2], [7], X is endowed with a fixed conformal structure, and one considers the space $M(X)$ of all conformal structures whose Teichmüller distance [6], [13] from the given one is finite. Let $Q_0(X)$ be the group of homeomorphisms of X , which are quasiconformal (relative to the given conformal structure) [6] and homotopic to the identity. $Q_0(X)$ operates on $M(X)$, and the quotient is $\mathbf{T}(X)$. Let $\Psi: M(X) \rightarrow \mathbf{T}(X)$ be the quotient map. Then Ψ does not define a fibre bundle with group $Q_0(X)$, for $Q_0(X)$ is not a topological group relative to the topology of $M(X)$. Still, Ψ is a locally trivial map [13], globally trivial if and only if $\mathbf{T}(X)$ is contractible. The Ahlfors-Bers theory applies to non-compact surfaces; it is not known in general whether $\mathbf{T}(X)$ is contractible.

6. We should verify that for compact X the Teichmüller space of Ahlfors and Bers coincides with ours. It is clear that there are a continuous injection $j: \mathbf{M}(X) \rightarrow M(X)$, and an open map $Q: M(X) \rightarrow \mathcal{S}$ satisfying $P = Q \circ j$, whose image is the classical Teichmüller space [3], [6]. That P and Q have the same image follows, for instance, from [11, Theorem 3]; the point is simply that every homeomorphism of X is homotopic to a diffeomorphism.

(E) We shall now outline an alternative proof that the action of $\mathbf{D}_0(X)$ on $\mathbf{M}(X)$ produces a trivial fibre bundle. Our proof makes essential use of an unpublished theorem of J. Sampson.

Each complex structure on X gives rise to a holomorphic covering map $\pi: U \rightarrow X$, and the hyperbolic metric on U thereby induces a metric on X of constant curvature -4 . Therefore we may interpret $\mathbf{M}(X)$ as the space of Riemannian metric structures of curvature -4 on X .

Given the metrics μ, ν in $\mathbf{M}(X)$ and a smooth map $f: X \rightarrow X$ we form its Dirichlet integral (energy)

$$E(f) = \frac{1}{2} \int_X \rho^2(f(z)) (|f_z|^2 + |f_{\bar{z}}|^2) dx dy .$$

Here $z = x + iy$ is an isothermal parameter relative to μ , and ν is given in isothermal parameters by $ds = \rho(w) |dw|$.

It was proved by Sampson and Eells [16] and by Shibata [28] that there is a smooth map $f: X \rightarrow X$ which has minimal energy among maps homotopic to the identity. (Such an f is called a *harmonic map*, relative to μ and ν .) The strictly negative curvature of ν and the formula for the second variation of E imply that the harmonic f is unique; we denote it by $f(\mu, \nu)$. Shibata [28] proved that $f(\mu, \nu)$ is a homeomorphism. Theorems of Lewy [25] and Heinz

[20] imply that f is a diffeomorphism. Thus, for any fixed μ in $\mathbf{M}(X)$, we obtain a map $\nu \mapsto f(\mu, \nu)$ from $\mathbf{M}(X)$ into $\mathbf{D}_0(X)$. Sampson has proved that this map is continuous (oral communication).

Let (X, ν) denote the manifold X endowed with the Riemannian metric ν . Since the composite of a harmonic map and an isometry is harmonic, we obtain the commutative diagram, where $g \in \mathbf{D}_0(X)$:

$$\begin{array}{ccc} (X, \mu) & \xrightarrow{f(\mu, \nu)} & (X, \nu) \\ & \searrow f(\mu, \nu \cdot g) & \uparrow g \\ & & (X, \nu \cdot g) \end{array}$$

Thus

$$(8.1) \quad g \circ f(\mu, \nu \cdot g) = f(\mu, \nu) \quad \text{for all } g \in \mathbf{D}_0(X) .$$

We now define a map $F: \mathbf{M}(X) \rightarrow \mathbf{T}(X) \times \mathbf{D}_0(X)$ by

$$F(\nu) = (\Phi(\nu), f(\mu, \nu)^{-1}) ,$$

where of course $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X)$ is the quotient map. Sampson's theorem implies that F is continuous. Moreover, (8.1) yields

$$(8.2) \quad F(\nu \cdot g) = F(\nu) \cdot g \quad \text{for all } g \in \mathbf{D}_0(X) ,$$

where $\mathbf{D}_0(X)$ acts on $\mathbf{T}(X) \times \mathbf{D}_0(X)$ in the obvious way: $(\tau, f) \cdot g = (\tau, f \circ g)$. It follows that F is injective, for if $F(\nu) = F(\nu')$, then $\Phi(\nu) = \Phi(\nu')$, so $\nu' = \nu \cdot g$, $g \in \mathbf{D}_0(X)$. Thus, by (8.2),

$$F(\nu) = F(\nu') = F(\nu \cdot g) = F(\nu) \cdot g ,$$

so $g = 1$ and $\nu = \nu'$. That F is surjective and a homeomorphism now follows from the identity

$$(\Phi(\nu), g) = F(\nu) \cdot f(\mu, \nu) \circ g = F(\nu \cdot f(\mu, \nu) \circ g) ,$$

valid for all $g \in \mathbf{D}_0(X)$. We conclude at once, *without appealing to either Teichmüller's theorem or §§ 5–7*, that $\mathbf{D}_0(X)$ and $\mathbf{T}(X)$ are contractible Hausdorff spaces, and that $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X)$ a trivial fibre bundle with structure group $\mathbf{D}_0(X)$. In fact, (8.2) means that F defines a bundle equivalence between $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X)$ and the trivial bundle $\pi_1: \mathbf{T}(X) \times \mathbf{D}_0(X) \rightarrow \mathbf{T}(X)$. An explicit section $\sigma: \mathbf{T}(X) \rightarrow \mathbf{M}(X)$ of Φ is given by

$$\sigma(\Phi(\nu)) = F^{-1}(\Phi(\nu), 1) = \nu \cdot f(\mu, \nu) .$$

9. The sphere

In this section X will be the Riemann sphere. Hence $\mathbf{D}(X)$ and $\mathbf{D}_0(X)$ coincide.

(A) **Proposition.** $\mathbf{D}_0(X)$ is homeomorphic to $G_C \times \mathbf{D}_0(X; 0, 1, \infty)$, where G_C is the group of holomorphic automorphisms of the sphere, and $\mathbf{D}_0(X; 0, 1, \infty)$ denotes the subgroup of $\mathbf{D}_0(X)$ of elements holding $0, 1, \infty$ fixed.

Proof. The map $(A, f) \mapsto A \circ f$ from $G_C \times \mathbf{D}_0(X; 0, 1, \infty)$ to $\mathbf{D}_0(X)$ is continuous, because $\mathbf{D}_0(X)$ is a topological group. Moreover, it is bijective, the inverse map being $f \mapsto (A_f, A_f^{-1} \circ f)$, where A_f is the unique member of G_C taking $0, 1, \infty$ to $f(0), f(1), f(\infty)$. Finally, $f \mapsto A_f$ is continuous by compactness properties of holomorphic functions.

Remark. G_C has the rotation group $\text{SO}(3)$ as maximal compact subgroup, and hence as strong deformation retract.

(B) Define the charts h_1 and h_2 on X by stereographic projection from 0 and ∞ respectively. Each $J \in \mathbf{M}(X)$ gives rise to a pair of functions $\mu_1, \mu_2 \in C^\infty(C, \mathcal{A})$ related (compare (5.1)) by

$$(9.1) \quad \mu_2(f(z)\overline{f'(z)})/f'(z) = \mu_1(z), \quad z \in C - \{0\},$$

where $f = h_2 \circ h_1^{-1}: C - \{0\} \rightarrow C - \{0\}$ is the map $z \mapsto 1/z$.

Let $w_i: C \rightarrow C$ be the normalized solution of Beltrami's equation $w_{\bar{z}} = \mu_i w_z (i = 1, 2)$. Then $f^{-1} \circ w_2 \circ f = w_1$ because of (9.1). In other words,

$$w_J = h_1^{-1} \circ w_1 \circ h_1 = h_2^{-1} \circ w_2 \circ h_2 \in \mathbf{D}_0(X; 0, 1, \infty).$$

Of course w_J is the unique element of $\mathbf{D}_0(X; 0, 1, \infty)$ which is a holomorphic map from X with complex structure J to X with its usual complex structure.

Theorem. The map $J \mapsto w_J$ is a homeomorphism from $\mathbf{M}(X)$ onto $\mathbf{D}_0(X; 0, 1, \infty)$.

Proof. The map is clearly bijective, and is a homeomorphism by applying Theorem 3B to both w_1 and w_2 .

Corollary (Smale [29]). $\text{SO}(3)$ is a strong deformation retract of $\mathbf{D}(X)$.

10. The torus

In this section X is a torus, and x_0 is a point of X . Since our arguments are quite similar to those we have already given for $g > 1$, we shall omit many details.

(A) Fix a point x_0 in X , and mark X by choosing a pair of simple loops a and b , which generate $\pi_1(X; x_0)$, so that a crosses b from left to right at x_0 and there are no other intersections. Analogous to Lemma 4C we have the

Lemma. For each J in $\mathbf{M}(X)$ there is a unique (J -)holomorphic covering map $\pi: C \rightarrow X$ with cover group Γ such that

1) *The loop a determines the translation*

$$Az = z + 1 \quad \text{in } \Gamma .$$

2) *The loop b determines the translation*

$$Bz = z + \tau \quad \text{in } \Gamma, \quad \text{Im } \tau > 0 .$$

3) $\pi(x_0) = 0$.

Now choose $J_0 \in \mathbf{M}(X)$, and let $\pi: C \rightarrow X$ and Γ be determined by the lemma. As in § 5 A, there is an induced map $\pi^*: \mathbf{M}(X) \rightarrow \mathbf{M}(C)$ whose image is $\mathbf{M}(\Gamma)$, the space of Γ -invariant complex structures on C . Because of the simple form of Γ , the equation for Γ -invariance of $\mu \in \mathbf{M}(C)$ becomes

$$(10.1) \quad \mu \circ \gamma = \mu \quad \text{for all } \gamma \in \Gamma .$$

As before, we denote by $A^1(\Gamma)$ the Fréchet space of all $\mu \in C^\infty(C, C)$ which satisfy (10.1). The following assertions are proved in the same way as the corresponding ones in § 5A.

Proposition. $\mathbf{M}(\Gamma)$ is the convex open set in $A^1(\Gamma)$ consisting of the $\mu \in A^1(\Gamma)$ such that $\sup \{|\mu(z)|: z \in C\} < 1$, and $\pi^*: \mathbf{M}(X) \rightarrow \mathbf{M}(\Gamma)$ is a diffeomorphism.

Corollary. $\mathbf{M}(X)$ is a contractible complex analytic manifold modeled on a Fréchet space.

(B) Let $\mathbf{D}_0(\Gamma)$ be the centralizer of Γ in $\mathbf{D}(C)$, and $\mathbf{D}_0(\Gamma; 0)$ the subgroup fixing 0. As in § 5B, define $\pi_*: \mathbf{D}_0(\Gamma; 0) \rightarrow \mathbf{D}(X)$ by $\pi_*(f) \circ \pi = \pi \circ f$.

Proposition. $\pi_*: \mathbf{D}_0(\Gamma; 0) \rightarrow \mathbf{D}_0(X; x_0)$ is an isomorphism of topological groups.

We follow the reasoning of § 5B, with the Euclidean metric in place of the hyperbolic metric.

(C) Once again, the natural action of $\mathbf{D}_0(X; x_0)$ on $\mathbf{M}(X)$ is transferred by π to the action

$$(10.2) \quad \mu_f \cdot g = \mu_{f \circ g}$$

of $\mathbf{D}_0(\Gamma; 0)$ on $\mathbf{M}(\Gamma)$. Analogous to Propositions 5C and 5D we have the

Proposition. *The action $\mathbf{M}(\Gamma) \times \mathbf{D}_0(\Gamma; 0) \rightarrow \mathbf{M}(\Gamma)$ given by (10.2) is free, continuous, and proper.*

Corollary. *The natural action $\mathbf{M}(X) \times \mathbf{D}_0(X; x_0) \rightarrow \mathbf{M}(X)$ is free, continuous, and proper.*

(D) Define $P: \mathbf{M}(X) \rightarrow U$ by $P(J) = \tau$, where $Bz = z + \tau$ is determined by Lemma 10A. Composing P with the inverse of $\pi^*: \mathbf{M}(X) \rightarrow \mathbf{M}(\Gamma)$ produces a map, still called $P: \mathbf{M}(\Gamma) \rightarrow U$. Analogous to §§ 6B and C we have

Lemma 1. *Let $\tau_0 = P(0) \in U$. Then*

$$P(\mu) = w_\mu(\tau_0) \quad \text{for all } \mu \in \mathbf{M}(\Gamma) .$$

Proof. For any $\mu \in \mathbf{M}(\Gamma)$, $\pi_\mu = \pi \circ w_\mu^{-1}: C \rightarrow X$ is the covering map determined by Lemma 10A, and $\Gamma_\mu = w_\mu \Gamma w_\mu^{-1}$. In particular, $B_0 z = z + P(0)$ and $B_\mu(z) = z + P(\mu)$ are related by $B_\mu = w_\mu \circ B_0 \circ w_\mu^{-1}$.

Lemma 2. $P(\mu) = P(\nu)$ if and only if μ and ν are $\mathbf{D}_0(\Gamma; 0)$ -equivalent.

Proof. Because J_0 was arbitrary we may assume $\nu = 0$. By Lemma 1, $P(\mu) = P(0)$ if and only if w_μ commutes with $z \mapsto z + \tau_0$ and hence with Γ (for w_μ always commutes with $z \mapsto z + 1$). But 0 and μ are $\mathbf{D}_0(\Gamma; 0)$ -equivalent if and only if $\mu = \mu_f$ for some $f \in \mathbf{D}_0(\Gamma; 0)$, which, being normalized, can only be w_μ .

(E) **Proposition.** $P: \mathbf{M}(\Gamma) \rightarrow U$ is continuous and surjective. Further, $\sigma: U \rightarrow \mathbf{M}(\Gamma)$ defined by

$$\sigma(z) = \frac{\tau_0 - z}{z - \bar{\tau}_0}$$

is a holomorphic section of P .

Proof. The continuity of P is immediate from Theorem 3B. By (10.1), all constant maps $\lambda: U \rightarrow \mathcal{A}$ are Γ -invariant complex structures; these form the image $\sigma(U)$. To verify that $P \circ \sigma: U \rightarrow U$ is the identity map, note that

$$w_\lambda(z) = (1 + \lambda)^{-1}(z + \lambda \bar{z}) .$$

Corollary. $P: \mathbf{M}(X) \rightarrow U$ is an open map.

In fact, $P: \mathbf{M}(\Gamma) \rightarrow U$ maps each neighborhood of $0 \in \mathbf{M}(\Gamma)$ to a neighborhood of $\tau_0 \in U$. But $0 \in \mathbf{M}(\Gamma)$ corresponds to an arbitrary $J_0 \in \mathbf{M}(X)$.

Remark. The holomorphic section σ was discovered by Teichmüller. Teichmüller's theorem [6] gives a section $\sigma: \mathbf{T}(X) \rightarrow \mathbf{M}(X)$ for any compact X , taking its values in the space $\mathbf{M}(X)$ of bounded measurable complex structures. But if the genus of X is greater than one, Teichmüller's section is not continuous.

(F) **Theorem.** The quotient map $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X) = \mathbf{M}(X)/\mathbf{D}_0(X; x_0)$ defines a trivial principal fibre bundle, and $\mathbf{T}(X)$ is homeomorphic to U .

Corollary. $\mathbf{D}_0(X; x_0)$ is contractible. Thus every fibre bundle with structure group $\mathbf{D}_0(X; x_0)$ is topologically trivial.

Those assertions merely consolidate the results of §§ 10C, D, E.

(G) **Proposition.** The map $X \times \mathbf{D}_0(X; x_0) \rightarrow \mathbf{D}_0(X)$ defined by $(\tau, f) \mapsto \tau \circ f$ is a homeomorphism.

Proof. Write any $f \in \mathbf{D}(X)$ in the form $\tau_f \circ f_0$ where $f_0(x_0) = x_0$. f is homotopic to f_0 .

Corollary. $\mathbf{D}_0(X)$ has X as strong deformation retract. In particular, it is the identity component of $\mathbf{D}(X)$.

11. Non-orientable surfaces

A closed non-orientable surface X cannot have a complex structure, but one can still consider the space $\mathbf{M}(X)$ of conformal structures on X . Moreover,

for any conformal structure there always exists a universal covering map $\pi: \bar{X} \rightarrow X$ such that the cover transformations are conformal maps. Here \bar{X} is the sphere, Euclidean plane, or hyperbolic plane (with its usual conformal structure). The methods of the previous sections can thus be applied to the study of non-orientable surfaces, with only minor changes of details. We shall outline here the principal results. In all cases we find that the diffeomorphism group $\mathbf{D}(X)$ has the same homotopy groups as the homeomorphism group (Hamstrom [19]).

(A) If X is not the real projective plane or the Klein bottle, there is a covering map $\pi: U \rightarrow X$, whose cover group Γ consists of conformal automorphisms of U . There is of course an induced map $\pi^*: \mathbf{M}(X) \rightarrow \mathbf{M}(\Gamma)$, where $\mathbf{M}(\Gamma)$ is the space of Γ -invariant conformal structures. The equation for Γ -invariance takes a new form for orientation-reversing elements of Γ ; $\mu \in \mathbf{M}(U)$ is Γ -invariant if and only if

$$(11.1) \quad (\mu \circ \gamma) \bar{\gamma}' / \gamma' = \mu \quad \text{if } \gamma \in \Gamma \text{ is holomorphic,}$$

$$(11.2) \quad (\mu \circ \gamma) \bar{\gamma}_z / \gamma_z = \bar{\mu} \quad \text{if } \gamma \in \Gamma \text{ reverses orientation.}$$

Let $A^1(\Gamma)$ be the Fréchet space of $\mu \in C^\infty(U, \mathbf{C})$ which satisfy (11.1) and (11.2). Because of (11.2), $A^1(\Gamma)$ is a real but not a complex linear space. In fact, let $\Gamma_0 \subset \Gamma$ be the normal subgroup of orientation preserving (holomorphic) maps. Then $A^1(\Gamma_0)$ is the direct sum of $A^1(\Gamma)$ and $iA^1(\Gamma)$. Still we have

Proposition. $\mathbf{M}(\Gamma)$ is an open convex set in $A^1(\Gamma)$, and $\pi^*: \mathbf{M}(X) \rightarrow \mathbf{M}(\Gamma)$ is a diffeomorphism. In particular, $\mathbf{M}(X)$ is contractible.

(B) Mimicing the reasoning of § 5 we obtain the

Proposition. The natural action of $\mathbf{D}_0(X)$ on $\mathbf{M}(X)$ is free, proper, and continuous.

Here $\mathbf{D}_0(X)$ is the group of diffeomorphisms homotopic to the identity, and X is not the projective plane nor the Klein bottle. To complete the story for such X , we note that our construction in § 8 of a harmonic section $\sigma: \mathbf{M}(X)/\mathbf{D}_0(X) \rightarrow \mathbf{M}(X)$ did not require X to be oriented. Defining the Teichmüller space $\mathbf{T}(X) = \mathbf{M}(X)/\mathbf{D}_0(X)$ we have

Theorem. The quotient map $\Phi: \mathbf{M}(X) \rightarrow \mathbf{T}(X)$ defines a trivial principal fibre bundle.

Corollary. $\mathbf{T}(X)$ and $\mathbf{D}_0(X)$ are contractible. In particular, $\mathbf{D}_0(X)$ is connected.

(C) The real projective plane X is the quotient of the Riemann sphere $\mathbf{C} \cup \{\infty\}$ by the group Γ of order two generated by the antipodal map $\gamma(z) = -1/\bar{z}$. The space $\mathbf{M}(X)$ of conformal structures on X is diffeomorphic to $\mathbf{M}(\Gamma)$, the space of $\mu \in \mathbf{M}(\mathbf{C})$ satisfying (11.2). (Comparison with (9.1) reveals that each $\mu \in \mathbf{M}(\mathbf{C})$ which satisfies (11.2) is also smooth at ∞ .)

Similarly, the group $\mathbf{D}(X)$ of all diffeomorphisms of X is diffeomorphic to $\mathbf{D}(\Gamma)$, the centralizer (= normalizer) of Γ in $\mathbf{D}(\mathbf{C})$. As in § 9, let G_c be the

group of all conformal automorphisms of $C \cup \{\infty\}$. The intersection of G_C and $\mathbf{D}(\Gamma)$ is $\text{SO}(3)$, the group of rotations of the sphere. Let N_0 be the set (not a group) of f in $\mathbf{D}(\Gamma)$ with $f(0) = 0$ and $f_z(0)$ real and positive. Since for any f in $\mathbf{D}(\Gamma)$, $|f_z(0)| \geq |J_f(0)| > 0$ (where J_f is the Jacobian of f), we have the

Lemma. $\mathbf{D}(\Gamma)$ is homeomorphic to $\text{SO}(3) \times N_0$.

Proposition. The map $\mu \mapsto \mu_f = f_{\bar{z}}/f_z$ is a homeomorphism from N_0 onto $\mathbf{M}(\Gamma)$.

Proof. $\mu \mapsto \mu_f$ is clearly a continuous map into $\mathbf{M}(C)$. It takes its values in $\mathbf{M}(\Gamma)$ because each $f \in N_0$ commutes with γ . It is injective because if $\mu_f = \mu_g$, then $f \circ g^{-1} \in G_C \cap \mathbf{D}(\Gamma) = \text{SO}(3)$; the normalization at 0 makes $f = g$. Finally, we must exhibit a continuous inverse map from $\mathbf{M}(\Gamma)$ to N_0 . Given $\mu \in \mathbf{M}(\Gamma)$, let $w = w_\mu$ be the normalized solution of (3.2). $w \circ \gamma \circ w^{-1} = h$ is an orientation-reversing conformal involution of the sphere. Since w is normalized, h interchanges 0 and ∞ . Further, h has no fixed points. It follows that $h(z) = r/\bar{z}$, where $r = h(1) = w(-1) < 0$. Put

$$f_\mu = (-r)^{-1/2} |w_z(0)| w_z(0)^{-1} w.$$

Clearly, $f_\mu \in N_0$ and satisfies (3.2). Theorem 3B implies that the map $\mu \mapsto f_\mu$ is continuous.

Corollary. The group of diffeomorphisms of the real projective plane has $\text{SO}(3)$ as strong deformation retract.

(D) It remains to consider the Klein bottle. We take $X = C/\Gamma$, where Γ is generated by $Az = \bar{z} + 1/2$ and $Bz = z + i$; as usual, $\pi: C \rightarrow X$ is the natural map. The space of Γ -invariant conformal structures is

$$\mathbf{M}(\Gamma) = \{\mu \in \mathbf{M}(C) : \mu \circ A = \mu, \mu \circ B = \mu\}.$$

Let $\mathbf{D}_0(\Gamma)$ be the centralizer of Γ in $\mathbf{D}(C)$, and $\pi^*: \mathbf{D}_0(\Gamma) \rightarrow \mathbf{D}_0(X)$ the natural map. The kernel of π^* is the group of all real translations $z \mapsto z + t$, $t \in \mathbf{R}$. Let N_0 be the set (not a group) of f in $\mathbf{D}_0(\Gamma)$ such that the real part of $f(0)$ vanishes.

Proposition.

- (a) $\mathbf{D}_0(X)$ is homeomorphic to $\text{SO}(2) \times N_0$.
- (b) N_0 is homeomorphic to

$$\mathbf{M}_0(\Gamma) = \{\mu \in \mathbf{M}(\Gamma) : w_\mu \circ B = B \circ w_\mu\}.$$

- (c) Define $\sigma: \mathbf{R}^+ \rightarrow \mathbf{M}(\Gamma)$ by $\sigma(r) = (1-r)(1+r)^{-1}$. For $\mu \in \mathbf{M}(\Gamma)$, $w_{\sigma(r)}^{-1} \circ w_\mu$ commutes with B if and only if $w_\mu(i) = ri$.
- (d) The map $(r, \lambda) \mapsto \mu$, where $w_\mu = w_{\sigma(r)} \circ w_\lambda$, is a homeomorphism from $\mathbf{R}^+ \times \mathbf{M}_0(\Gamma)$ onto $\mathbf{M}(\Gamma)$.

The proofs, which we omit, are analogous to several others in §§ 10 and 11.

Corollary. Let X be the Klein bottle. Then $\mathbf{D}_0(X)$ has $\text{SO}(2)$ as strong deformation retract.

Remark. For every X except the projective plane and Klein bottle, we have found a subgroup G_0 of $\mathbf{D}(X)$ acting freely on $\mathbf{M}(X)$ such that the natural map from $\mathbf{M}(X)/G_0$, the Teichmüller space, onto $\mathbf{M}(X)/\mathbf{D}(X)$ is a ramified covering map. For the projective plane and Klein bottle, however, our luck ran out. We were compelled to use subsets N_0 of $\mathbf{D}(X)$ which were not subgroups. Alternatively, we could have chosen subgroups G_0 contained in N_0 , at the cost of accepting quotient space $\mathbf{M}(X)/G_0$ of higher dimension. For more general manifolds X and spaces of structures, of course, the unlucky cases are the rule. It seems very unusual to have a subgroup of $\mathbf{D}(X)$ which acts freely and produces a finite dimensional quotient.

References

- [1] L. V. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. of Math. **74** (1961) 171–191.
- [2] —, *Teichmüller spaces*, Proc. Internat. Congress Math. (1962), Institute Mittag-Leffler, Djursholm, Sweden, 1963, 3–9.
- [3] —, *Lectures on quasiconformal mappings*, Math. Studies No. 10, Van Nostrand, Princeton, 1966.
- [4] L. V. Ahlfors & L. Bers, *Riemann's mapping theorem for variable metrics*, Ann. of Math. **72** (1960) 385–404.
- [5] R. Arens, *Topologies for homeomorphism groups*, Amer. J. Math. **68** (1946) 593–610.
- [6] L. Bers, *Quasiconformal mappings and Teichmüller's theorem*, Analytic functions, Princeton University Press, Princeton, 1960, 89–119.
- [7] —, *Spaces of Riemann surfaces*, Proc. Internat. Congress Math., Edinburgh, 1958, 349–361.
- [8] —, *Automorphic forms and general Teichmüller spaces*, Proc. Conf. Complex Analysis (Minneapolis, 1964), Springer, Berlin, 1965, 109–113.
- [9] L. Bers & L. Ehrenpreis, *Holomorphic convexity of Teichmüller spaces*, Bull. Amer. Math. Soc. **70** (1964) 761–764.
- [10] L. Bers, F. John & M. Schechter, *Partial differential equations*, Interscience, New York, 1964.
- [11] C. J. Earle, *Reduced Teichmüller spaces*, Trans. Amer. Math. Soc. **126** (1967) 54–63.
- [12] —, *On holomorphic cross-sections in Teichmüller spaces*, to appear.
- [13] C. J. Earle & J. Eells, Jr., *On the differential geometry of Teichmüller spaces*, J. Analyse Math. **19** (1967) 35–52.
- [14] —, *The diffeomorphism group of a compact Riemann surface*, Bull. Amer. Math. Soc. **73** (1967) 557–559.
- [15] C. J. Earle & A. Schatz, *Teichmüller theory for surfaces with boundary*, to appear.
- [16] J. Eells, Jr. & J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964) 109–160.
- [17] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. **115** (1966) 83–107.
- [18] A. Grothendieck, *Techniques de construction en géométrie analytique*, Séminaire H. Cartan, Paris, 1960–61, Exp. 7–8.
- [19] M. E. Hamstrom, *Homotopy groups of the space of homeomorphisms on a 2-manifold*, Illinois J. Math. **10** (1966) 563–573.
- [20] E. Heinz, *Über gewisse elliptische Systeme von Differentialgleichungen*, Math. Ann. **131** (1956) 411–428.
- [21] D. Husemoller, *Fibre bundles*, McGraw-Hill, New York, 1966.

- [22] K. Kodaira & D. C. Spencer, *On deformations of complex analytic structures I, II*, Ann. of Math. **67** (1958) 328–466.
- [23] S. Kravetz, *On the geometry of Teichmüller spaces and the structure of their modular groups*, Ann. Acad. Sci. Fenn. **278** (1959) 1–35.
- [24] O. Lehto & K. I. Virtanen, *Quasikonforme Abbildungen*, Springer, Berlin, 1965.
- [25] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936) 689–692.
- [26] H. E. Rauch, *A transcendental view of the space of algebraic Riemann surfaces*, Bull. Amer. Math. Soc. **71** (1965) 1–39.
- [27] J. P. Serre, *Un théorème de dualité*, Comment. Math. Helv. **29** (1955) 9–26.
- [28] K. Shibata, *On the existence of a harmonic mapping*, Osaka Math. J. **15** (1963) 173–211.
- [29] S. Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959) 621–626.
- [30] O. Teichmüller, *Extremale quasikonforme Abbildungen und quadratische Differentiale*, Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **22** (1939)
- [31] A. Weil, *Variétés Kähleriennes*, Actualités Sci. Indust. No. 1267, 1958.
- [32] —, *Modules des surfaces de Riemann*, Séminaire Bourbaki, 11 ième année, 1958–59, Exp. 168.
- [33] —, *Remarks on the cohomology of groups*, Ann. of Math. **80** (1964) 149–157.
- [34] H. Weyl, *Die Idee der Riemannschen Fläche*, Teubner, Leipzig, 1955.

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